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International Journal of Solids and Structures 42 (2005) 1465–1490
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

A generalized, multilength scale framework for thermo-diffusional-mechanically coupled, nonlinear, laminated plate theories with delaminations

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Received 13 May 2004; received in revised form 5 August 2004
Available online 5 November 2004

Abstract

A new type of plate theory based on a general, unified, theoretical framework for the response of (von Karman) nonlinear, delaminated plate theories in the presence of thermo-diffusional-mechanical coupling is presented. The theory is based on the unique use of two length scale expansions obtained from a superposition of global and local effects for the displacement, temperature, and solute concentration fields. The orders and forms of these local and global displacement, temperature, and solute fields are arbitrary. The theory incorporates delamination and/or nonlinear elastic or inelastic interfacial behavior for the mechanical, thermal, and concentration effects in a unified fashion through the use of generalized interfacial constitutive (cohesive) relations. The mathematical framework introduces new types of coupling effects between the different length scale effects of all three fields. The resulting unified theoretical framework can be used to consider the general thermo-diffusionally-mechanically coupled response of laminated (or homogeneous) plates in the presence of delaminations, buckling, and/or nonlinear material behavior. The author is unaware of any previous attempts to develop plate theory formulations capable of considering the multitude of effects incorporated into the proposed framework.

It is shown that existing displacement-based plate theories for both the mechanical as well as thermo-mechanical behavior of laminated plates can be obtained via suitable specializations of the proposed framework. New types of plate theories can be obtained through various specializations of the proposed general theory.

Keywords: Laminated plates; Unified plate theory; Multilength scale plate theory; HSDT; Thermo-diffusional-mechanical coupling; Diffusion; Hygrothermoelastic; Delamination; Interfacial constitutive relations; Geometric nonlinearity

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doi:10.1016/j.ijsolstr.2004.08.007

1. Introduction

The design of advanced structures has seen the increased usage of laminated (composite) plates and shells. As the design demands on such structural members increase it becomes more important that accurate theories be used in the analysis of such structural components. A natural choice for such analyses are two-dimensional (2D) plate/shell theories. These types of theories provide a potentially more efficient and accurate alternative to the use of three-dimensional finite element (FE) analysis.

To correctly model the response of laminated structures a 2D plate/shell theory must incorporate a variety of different deformation mechanisms. These mechanisms occur at several characteristic length-scales. These length scales can be roughly separated into three categories.

The first such length scale occurs at the micro-level and is associated with material microstructure (i.e. fiber/matrix in a composite material or metal grains within a polycrystal). Typically, the response at this length scale is considered within the context of constitutive models. Given the many forms of such constitutive relations a plate/shell theory should be sufficiently general as to be able to incorporate any type of material constitutive model. Furthermore, given the nonlinear evolution equations present in general constitutive models, a plate/shell theory must be able to accurately predict the local fields in order to correctly predict the plate response.

The next higher length scale (the meso-length scale) is associated with the lamina thickness. A primary deformation mechanism associated with this characteristic length is delamination. Delaminations typically develop during service and can significantly reduce the loading carrying capabilities of a laminated structure. These considerations imply that plate and shell theories should be able to predict the initiation and growth of delaminations at any point (and at any time) in a structure.

The final category of characteristic length scales, the global length scale, is associated with the entire laminate thickness. Typical responses associated with this length scale are the buckling or gross motion of the entire plate/shell. Correct modeling of these types of behavior requires that geometric nonlinear effects be incorporated into structural theories.

While the above effects have been discussed individually, it must be recognized that these mechanisms are, typically, interactive. For example, it has been shown that viscoplasticity can have a significant effect on the critical bucking load and the post-behavior of laminated plates (Gilat and Aboudi, 1994). Other examples of interactive effects exist throughout the literature. The presence of these interactive effects implies the need for structural theories that are capable of coupling different phenomena over different length scales.

The previous discussion has centered around the influence of different mechanical effects on the behavior of plates/shells. The behavior of a plate/shell can also be influenced by the presence of temperature and/or solute concentration fields.

The temperature and deformation fields are always coupled, thermo-mechanical coupling (TMC), due to the presence of both temperature and deformation effects in the constitutive, equilibrium, and energy equations of a continuum. A basic presentation of the issues associated with TMC phenomena is given by Boley and Wiener (1985).

Thermo-mechanical coupling impacts the structural response at all three of the length scales discussed above. At the micro-level, the presence of history-dependent phenomena in the form of the inelastic work term in the energy equation results in a potentially strong forcing term for the local temperature rate. This type of effect in monolithic materials was considered by Allen (1991) where it was shown that TMC can cause significant deviations from an uncoupled thermo-mechanical analysis. Recent work by Williams and Aboudi (1999) considered the influence of TMC effects on the constitutive behavior of composites. It was shown that the influence of TMC at the microstructural level on the effective constitutive behavior of particulate composites could lead to significant deviations in the predicted effective stress–strain response as compared to an uncoupled analysis.

At the meso-scale coupling between delaminations and temperature could take several forms. In particular, the energy dissipation due to fracture/delamination and/or subsequent frictional sliding of

the separated surfaces can generate localized heating in the adjacent materials. Additionally, the presence of delaminations can result in distinctly different temperature distributions (as compared to perfectly bonded structures) since the presence of distinct cracks prevents heat transfer across interfaces.

At the global length scale, the work by Gilat and Aboudi (1996) on the buckling behavior of laminated and homogeneous plates has shown that TMC effects can significantly alter both the critical and post-buckling response of composite plates as compared to the predictions obtained from an uncoupled analysis. This work also gives a partial review of previous modeling which has considered the impact of TMC on the response of plates.

The presence of solute concentrations within a plate can impact the plate behavior in a fashion analogous to that of temperature effects due to the fact that, in general, the temperature, deformation, and solute concentration fields can be considered to be coupled through the constitutive, equilibrium, energy, and solute conservation equations of a continuum (Weitsman, 1987; Sih et al., 1986). This type of coupling is referred to as thermo-diffusional-mechanical coupling (TDMC) or as stress-assisted diffusion.

Since advanced structures operate under service conditions where TDMC effects can be significant it is important that plate/shell theories be able to consider TDMC effects. Relatively little work seems to have been carried out considering the impact of full coupling between thermal and mechanical effects compared to the amount of work done on the mechanical response of plates. To date there does not seem to exist a general theoretical treatment of TDMC within the context of plate/shell theories.

The purpose of this paper is to present a generalized, unified, theoretical framework for the development of any order/type of multilength scale, (von Karman) nonlinear, laminated, plate theory in the presence of thermo-diffusional-mechanical coupling and delamination. The theory is based on the use of generalized expansions composed of global and local effects for the displacement, temperature, and solute concentration fields within the laminated plate. The use of multilength scale field expansions introduces unique coupling effects between the global and local scales in all of the fields. The thermo-diffusional-mechanically coupled governing equations for the behavior of the plate are developed using Hamilton's principle and the method of moments form of the energy equation, and the equation for the conservation of solute. The effects of delamination on the different fields are introduced into the theory in a internally consistent fashion through the use of generalized interfacial constitutive models (ICMs). Geometric nonlinearity is introduced into the theoretical framework through the use of Von Karman type strains. No restrictions are imposed on the material constitutive relations for the layers. The resulting governing equations are functions of the fundamental unknown kinematic, temperature, and solute variables used in the expansions for the different fields in the individual layers. The proposed theoretical framework is an generalization of the purely mechanical theory presented by Williams (1999).

There are several aspects of the theory that are significance: First, the theory represents a new type of plate theory capable of considering thermo-mechanical-diffusional and delamination phenomena simultaneously. The second unique aspect is the multiscale nature of the theory obtained from the superposition of generalized local and global fields. Third, the theory seems to be the first plate theory capable of considering this broad range of effects in a coupled fashion. Furthermore, since any order/type of multilength scale theory or any currently available displacement based (''smeared'', discrete layer, ''zig-zag'', or homogeneous) plate theory can be obtained as a subset of the proposed framework, the current approach represents a unified framework for the development of plate theories.

2. Theoretical framework

The following conventions are used throughout the formulation. Superscripts (k) denote the layer number. Otherwise, superscripts will denote the number (order) of both global and local functions. Subscripts denote tensorial indices. Greek subscripts are considered to have a range of 1 and 2. Latin subscripts have a

range of 1–3. Repeated superscripts, (other than (k)), and subscripts imply summation. A comma or ∂_i denotes differentiation with respect to the spatial coordinates, x_i. A dot denotes differentiation with respect to time.

The global coordinates associated with the plate are x_1, x_2, x_3. The reference plane ($x_3 = 0$) is taken at the midplane of the laminate. The plate is considered to be composed of an arbitrary number of layers, N. A layer can contain a single lamina, several lamina, or a sublamina region. The local coordinates for a layer are denoted by x_α, $\bar{x}_3$ where $x_3^{(k)} \leqslant \bar{x}_3 \leqslant x_3^{(k+1)}$, Fig. 1.

2.1. Field expansions

The displacement field for the kth layer is given by

$$\begin{aligned} u_\alpha^{(k)}(\underline{x},t) &= U_\alpha^r(x_\alpha,t)P^r(x_3) + \mu_\alpha^{(k)s}(x_\alpha,t)p^{(k)s}(\bar{x}_3) \\ u_3^{(k)}(\underline{x},t) &= U_3^p(x_\alpha,t)\Pi^p(x_3) + \mu_3^{(k)q}(x_\alpha,t)\pi^{(k)q}(\bar{x}_3) \end{aligned} \tag{2.1}$$

where $r = \overline{J}_{\min},\ldots,\overline{J}_{\max}$ and $s = \widehat{J}_{\min},\ldots,\widehat{J}_{\max}$ and $p = \overline{L}_{\min},\ldots,\overline{L}_{\max}$ and $q = \widehat{L}_{\min},\ldots,\widehat{L}_{\max}$. Eq. (2.1) represents a generalization of the original displacement expansions used by Williams (1999) since it allows for different order expansions in the inplane and out of plane displacement components.

The temperature field within the kth layer is assumed to be of the following form

$$T^{(k)}(\underline{x},t) = \Theta^a(x_\alpha,t)G^a(x_3) + \theta^{(k)b}(x_\alpha,t)g^{(k)b}(\bar{x}_3) \tag{2.2}$$

where $a = \bar{A}_{\min},\ldots,\bar{A}_{\max}$ and $b = \overline{B}_{\min},\ldots,\overline{B}_{\max}$.

The solute concentration field within the kth layer is given by

$$m^{(k)}(\underline{x},t) = S^c(x_\alpha,t)L^c(x_3) + s^{(k)d}(x_\alpha,t)l^c(x_3) \tag{2.3}$$

where $c = \overline{C}_{\min},\ldots,\overline{C}_{\max}$ and $d = \overline{D}_{\min},\ldots,\overline{D}_{\max}$.

The functional forms and the orders of the expansions for the displacement, temperature, and solute fields are arbitrary and, thus, the functions P^r, Π^p, G^a, L^c, $p^{(k)s}$, $\pi^{(k)q}$, $g^{(k)b}$, and $l^{(k)d}$ are unspecified. The only restriction on the specification of the different components of the fields is that the expansion functions for a given component of a field must be independent. A mathematically rigorous methodology for obtaining the necessary independence of field components is to require that the functions be orthogonal.

$$\begin{aligned} &\int_{x_3^{(k)}}^{x_3^{(k+1)}} P^m p^{(k)n}\,\mathrm{d}\bar{x}_3 = 0 \\ &\int_{x_3^{(k)}}^{x_3^{(k+1)}} \Pi^m \pi^{(k)n}\,\mathrm{d}\bar{x}_3 = 0 \\ &\int_{x_3^{(k)}}^{x_3^{(k+1)}} G^m g^{(k)n}\,\mathrm{d}\bar{x}_3 = 0 \\ &\int_{x_3^{(k)}}^{x_3^{(k+1)}} L^m l^{(k)n}\,\mathrm{d}\bar{x}_3 = 0 \end{aligned} \tag{2.4}$$

The actual application of Eq. (2.4) is dependent upon the particular forms chosen for the expansion functions P^r, Π^p, G^a, L^c, $p^{(k)s}$, $\pi^{(k)q}$, $g^{(k)b}$, and $l^{(k)d}$. In choosing the local functional form it must be noted that an independent local constant term cannot be used since it is impossible to orthogonalize two constants, i.e. global and local constant terms over the layer thickness. Alternative possibilities for ensuring independence of the expansion functions are discussed more fully by Williams (1999). The local parts of the expansions in the different layers can potentially be of different order/form.

The layer displacement, temperature, and solute fields are functions of two different length scales with respect to the thickness direction in the plate. The first set of terms in the displacement (U_α^r and

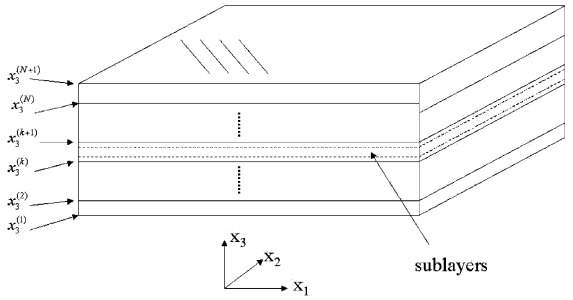

Fig. 1. Plate geometry and local layer coordinates.

U_3^p), temperature (Θ^a), and solute (S^c) fields, are associated with the global or structural length scale variation in the fields. The associated expansion functions P^r, Π^p, G^a, and L^c are continuous functions of the plate thickness coordinate x_3 for $-\frac{h}{2} \leqslant x_3 \leqslant \frac{h}{2}$ where h is the laminate thickness. The second set of terms in the displacement, temperature, and solute fields; $\mu_\alpha^{(k)s}$, $\mu_3^{(k)q}$, $\theta^{(k)b}$ and $s^{(k)d}$, represent variations in the global fields, i.e. local effects. The associated local expansion functions $p^{(k)s}$, $\pi^{(k)q}$, $g^{(k)b}$, and $l^{(k)d}$ are dependent only on the local coordinates within the kth layer and are identically zero outside this layer. Both the global and local components of the different fields vary simultaneously within the kth layer.

There are a number of physical interpretations that can be assigned to the assumed forms for the displacements, temperature, and solute concentration fields, Eqs. (2.1)–(2.3). The global components of the fields account for the gross behavior or overall trends in the response of the plate. The local field effects can be interpreted as the variations in the global fields induced by the presence of local microstructure caused by adjacent lamina with different fiber orientations, delaminations, or other meso-scale structural effects/inhomogeneities. Alternatively, the local fields could be considered as the necessary corrections to the global fields required to obtain accurate predictions for different loading situations in homogeneous plates, for example dynamic loading situations (wave propagation effects).

2.2. Strain–displacement relations

To account for moderate rotations in the structure, such as due to buckling, geometric nonlinearities are introduced into the theory through the use of von Karman type strains. Substituting Eq. (2.1) into the von Karman type strain assumptions results in the following strain field within the kth layer

$$
\begin{aligned}
\epsilon_{11}^{(k)} &= \partial_1 U_1^r P^r + \partial_1 \mu_1^{(k)s} p^{(k)s} + \frac{1}{2}\left(\partial_1 U_3^p \Pi^p + \partial_1 \mu_3^{(k)q} \pi^{(k)q}\right)^2 \\
\epsilon_{22}^{(k)} &= \partial_2 U_2^r P^r + \partial_2 \mu_2^{(k)s} p^{(k)s} + \frac{1}{2}\left(\partial_2 U_3^p \Pi^p + \partial_2 \mu_3^{(k)q} \pi^{(k)q}\right)^2 \\
\epsilon_{33}^{(k)} &= U_3^p \partial_3 \Pi^p + \mu_3^{(k)q} \partial_3 \pi^{(k)q} \\
\epsilon_{23}^{(k)} &= \frac{1}{2}\left(U_2^r \partial_3 P^r + \mu_2^{(k)s} \partial_3 p^{(k)s} + \partial_2 U_3^p \Pi^p + \partial_2 \mu_3^{(k)q} \pi^{(k)q}\right) \\
\epsilon_{13}^{(k)} &= \frac{1}{2}\left(U_1^r \partial_3 P^r + \mu_1^{(k)s} \partial_3 p^{(k)s} + \partial_1 U_3^p \Pi^p + \partial_1 \mu_3^{(k)q} \pi^{(k)q}\right) \\
\epsilon_{12}^{(k)} &= \frac{1}{2}\left[\partial_2 U_1^r P^r + \partial_2 \mu_1^{(k)s} p^{(k)s} + \partial_1 U_2^r P^r + \partial_1 \mu_2^{(k)s} p^{(k)s}\right] \\
&\quad + \frac{1}{2}\left[\left(\partial_1 U_3^p \Pi^p + \partial_1 \mu_3^{(k)q} \pi^{(k)q}\right)\left(\partial_2 U_3^p \Pi^p + \partial_2 \mu_3^{(k)q} \pi^{(k)q}\right)\right]
\end{aligned}
\tag{2.5}
$$

The results given in Eq. (2.5) are different than those given by Williams (1999) due to the use of the (potentially) different expansions orders in Eq. (2.1).

2.3. Equations of motion

The equations of motion are obtained using Hamilton's principle

$$\int_{t_1}^{t_2}\left(\int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V - \int_V \rho\ddot{u}_i\delta u_i\,\mathrm{d}V - \int_V f_i\delta u_i\,\mathrm{d}V - \int_{\partial V} t_i\delta u_i\,\mathrm{d}S\right)\mathrm{d}t = 0 \tag{2.6}$$

where $V = h\Omega$, Ω is an arbitrary inplane area on the reference surface and ∂V is the outer surface of V. In Eq. (2.6) the σ_{ij} denote stresses, the ϵ_{ij} denote (nonlinear) strains, the t_i are surface tractions, the body forces are given by f_i, and ρ is the density.

Using Eqs. (2.1) and (2.5) in Eq. (2.6) the following equations of motion are obtained:

$$\begin{aligned}
&\overline{I}^{rj}\ddot{U}^r_\alpha + \sum_{k=1}^{N}\widehat{I}^{(k)sj}\ddot{\mu}^{(k)s}_\alpha = \overline{N}^j_{\alpha\beta,\beta} - \overline{R}^j_{\alpha 3} + \overline{F}^j_\alpha + \overline{\tau}^j_\alpha\\
&\breve{I}^{pj}\ddot{U}^p_3 + \sum_{k=1}^{N}\breve{I}^{(k)qj}\ddot{\mu}^{(k)q}_3 = \breve{N}^j_{3\beta,\beta} - \breve{R}^j_{33} + \partial_\alpha\left(\breve{M}^{pj}_{\alpha\beta}U^p_{3,\beta} + \sum_{k=1}^{N}M^{(k)qj}_{\alpha\beta}\mu^{(k)q}_{3,\beta}\right) + \breve{F}^j_3 + \breve{\tau}^j_3\\
&\widehat{I}^{(k)jr}\ddot{U}^r_\alpha + \widetilde{I}^{(k)sj}\ddot{\mu}^{(k)s}_\alpha = \widehat{N}^{(k)j}_{\alpha\beta,\beta} - \widehat{R}^{(k)j}_{\alpha 3} + \widehat{F}^{(k)j}_\alpha + \hat{\tau}^{(k)j}_\alpha\\
&\breve{I}^{(k)jp}\ddot{U}^p_3 + I^{*(k)qj}\ddot{\mu}^{(k)q}_3 = \breve{N}^{(k)j}_{3\beta,\beta} - \breve{R}^{(k)j}_{33} + \partial_\alpha\left(M^{(k)jp}_{\alpha\beta}U^p_{3,\beta} + \breve{M}^{(k)jq}_{\alpha\beta}\mu^{(k)q}_{3,\beta}\right) + \breve{F}^{(k)j}_3 + \breve{\tau}^{(k)j}_3
\end{aligned} \tag{2.7}$$

where $j = \overline{J}_{\min},\ldots,\overline{J}_{\max}$ in Eq. (2.7a), $j = \overline{L}_{\min},\ldots,\overline{L}_{\max}$ in Eq. (2.7b), $j = \widehat{J}_{\min},\ldots,\widehat{J}_{\max}$ in Eq. (2.7c), $q = \widehat{L}_{\min},\ldots,\widehat{L}_{\max}$ in Eq. (2.7d), $k = 1,\ldots,N$, and N is the number of layers in the laminate. The definitions for the resultants in Eq. (2.7) are given in Appendix A. Eqs. (2.7a and b) are the equations of motion associated with the global displacement effects U^r_α and U^p_3. Thus, these equations can be considered to be "smeared" equations of motion for the plate. Conversely, Eqs. (2.7c and d) are related to the local effects $\mu^{(k)s}_\alpha$ and $\mu^{(k)q}_3$ and, thus, can be interpreted as local equations of motion. While the different types of equations of motion are related to either a global or local displacement effect they are influenced by all of the different length scale effects of all of the different fields. This will be discussed in more detail later. The above system of equations could have been obtained from an equivalent vectorial (method of moments) analysis as shown by Soldatos (1995) or Gilat (1996, 1998) where the weighting functions are the expansion functions rather than simple polynomial terms in x_3.

The essential boundary conditions obtained from the variational analysis are given by the specification of

$$U^j_i, \quad \mu^{(k)j}_i \quad \text{on } \partial\Omega_1 \tag{2.8}$$

where superscript j takes the appropriate range for the displacement component. The associated natural boundary conditions are given by

$$\begin{aligned}
&\overline{T}^j_\alpha = \overline{N}^j_{\alpha\beta}n_\beta\\
&\breve{T}^j_3 = \left[\breve{N}^j_{3\alpha} + \left(\breve{M}^{rj}_{\alpha\beta}\partial_\beta U^r_3 + \sum_{k=1}^{N}M^{(k)sj}_{\alpha\beta}\partial_\beta\mu^{(k)s}_3\right)\right]n_\alpha\\
&\widehat{T}^{(k)j}_\alpha = \widehat{N}^{(k)j}_{\alpha\beta}n_\beta\\
&\breve{T}^{(k)j}_3 = \left[\breve{N}^{(k)j}_{3\alpha} + \left(M^{(k)jr}_{\alpha\beta}\partial_\beta U^r_3 + \breve{M}^{(k)sj}_{\alpha\beta}\partial_\beta\mu^{(k)s}_3\right)\right]n_\alpha \quad \text{on } \partial\Omega_2
\end{aligned} \tag{2.9}$$

The boundary of Ω is denoted by $\partial\Omega$ where $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$. The definitions used in Eq. (2.9) are given in Appendix A.

The governing equations of motion associated with the individual layers, Eqs. (2.7c) and (2.7d), are coupled through the interfacial constraints. In the presence of delaminations the appropriate forms for the displacement, temperature, and solute jumps across the kth interface are

$$\begin{aligned}
[u_\alpha]^{(k)} &= \left(\mu_\alpha^s p^s(x_3^{(k+1)})\right)^{(k+1)} - \left(\mu_\alpha^s p^s(x_3^{(k+1)})\right)^{(k)} \\
[u_3]^{(k)} &= \left(\mu_3^q \pi^q(x_3^{(k+1)})\right)^{(k+1)} - \left(\mu_3^q \pi^q(x_3^{(k+1)})\right)^{(k)} \\
[T]^{(k)} &= \left(\theta^b g^b(x_3^{(k+1)})\right)^{(k+1)} - \left(\theta^b g^b(x_3^{(k+1)})\right)^{(k)} \\
[m]^{(k)} &= \left(s^d l^d(x_3^{(k+1)})\right)^{(k+1)} - \left(s^d l^d(x_3^{(k+1)})\right)^{(k)}
\end{aligned} \tag{2.10}$$

If there are no delaminations then $[u_i]^{(k)} = 0$, $[T]^{(k)} = 0$, and $[m]^{(k)} = 0$.

The corresponding traction continuity conditions are

$$(t_i)^{(k)}\Big|_{x_3^{(k+1)}} = -(t_i)^{(k+1)}\Big|_{x_3^{(k+1)}} \tag{2.11a}$$

or, alternatively,

$$\sigma_{i3}^{(k+1)}\Big|_{x_3^{(k+1)}} = \sigma_{i3}^{(k)}\Big|_{x_3^{(k+1)}} = \sigma_{i3}^{(k)I} \tag{2.11b}$$

where $\sigma_{i3}^{(k)I}$ are defined to be the interfacial stresses at the kth interface. If there are no discontinuities in the field variables then the above results, Eqs. (2.1)–(2.11), represent the mechanical part of the necessary theoretical framework for the analysis of perfectly bonded plates.

Delamination initiation and growth can be modeled through the use of interfacial constitutive models (ICMs) without any a priori assumptions about the size or location of potential delaminations. The mechanical forms of these models relate the field jumps across an interface, Eq. (2.10), to the corresponding interfacial tractions, Eq. (2.11). The general functional form for these models is

$$\mathbf{t} = \mathbf{f}([\mathbf{u}], [T], [m]) \tag{2.12}$$

where the vector $\mathbf{t}$ represents the interfacial tractions, $[\mathbf{u}]$ is the vector of displacement jumps across the interface, and $[T]$ is the temperature jump across the interface. Typical response curves for both the normal (t_n) and shear response (t_s) of a generic mechanical ICM are shown in Fig. 2. In general, the shear and normal responses in a mechanical ICM are nonlinear and coupled. The initial slope of the response curves can be related to the stiffness and thickness of the interlaminar region (Aboudi, 1991). After the peak stress is reached the interface can exhibit softening. The normal response exhibits closure under compression (no interpenetration) while the shear response is anti-symmetric. Interfacial constitutive models have a direct relation to classical fracture mechanics (Corigliano, 1993). Various proposed forms for mechanical ICMs are given by Aboudi (1991), Corigliano (1993), Needleman (1987, 1990), and McGee and Herakovich (1992).

Use of mechanical ICMs allows the interfacial tractions σ_{i3} to be eliminated from the formulation. Combining Eqs. (2.10)–(2.12) results in the unknown interfacial stresses being expressed as functions of the unknown kinematic, thermal, and solute terms in the different expansions, Eqs. (2.1)–(2.3), for the $k+1$th and kth layers ($\sigma_{i3}^{(k)I} = f_i^{(k)}([\underline{u}]^{(k)}, [T]^{(k)}), [m]^{(k)})$. Substituting this expression into Eqs. (2.7c) and (2.7d) eliminates the interfacial tractions from the equations of motion.

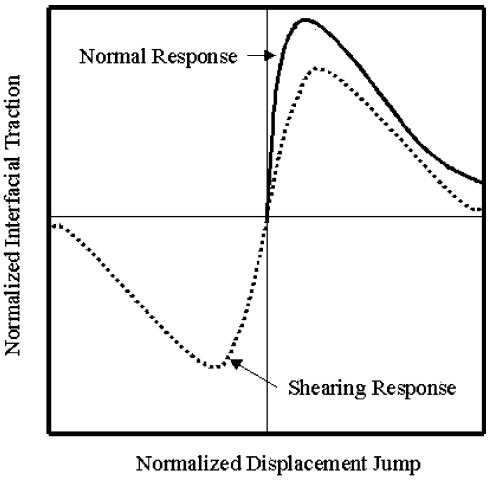

Fig. 2. Typical response curves for both the normal (t_n) and shear response (t_s) of a generic mechanical ICM.

$$\begin{aligned}
&\widehat{I}^{(k)jr}\ddot{U}^r_\alpha + \widetilde{I}^{(k)sj}\ddot{\mu}^{(k)s}_\alpha = \widehat{N}^{(k)j}_{\alpha\beta,\beta} - \widehat{R}^{(k)j}_{\alpha 3} + \widehat{F}^{(k)j}_\alpha + \Delta^{(k)j}_\alpha \\
&\check{I}^{(k)jr}\ddot{U}^r_3 + I^{*(k)sj}\ddot{\mu}^{(k)s}_3 = \check{N}^{(k)j}_{3\beta,\beta} - \check{R}^{(k)j}_{33} + \partial_\alpha\left(M^{(k)jr}_{\alpha\beta}U^r_{3,\beta} + \check{M}^{(k)js}_{\alpha\beta}\mu^{(k)s}_{3,\beta}\right) + \check{F}^{(k)j}_3 + \Delta^{(k)j}_3
\end{aligned} \tag{2.13}$$

where

$$\Delta^{(k)j}_\alpha = \begin{cases} f_\alpha\left([\mathbf{u}]^{(1)}, [T]^{(1)}, [m]^{(1)}\right)p^{(1)j}\left(x^{(2)}_3\right) - \sigma_{\alpha 3}\left(x^{(1)}_3\right)p^{(1)j}\left(x^{(1)}_3\right) & \text{for } k = 1 \\ f_\alpha\left([\mathbf{u}]^{(k+1)}, [T]^{(k+1)}, [m]^{(k+1)}\right)p^{(k)j}\left(x^{(k+1)}_3\right) - f_\alpha\left([\mathbf{u}]^{(k)}, [T]^{(k)}, [m]^{(k)}\right)p^{(k)j}(x^{(k+1)}_3) & \\ \quad \text{for } k = 2, \ldots, N-1 & \\ \sigma_{\alpha 3}\left(x^{(N+1)}_3\right)p^{(N+1)j}\left(x^{(N+1)}_3\right) - f_\alpha\left([\mathbf{u}]^{(N-1)}, [T]^{(N-1)}, [m]^{(N-1)}\right)p^{j(N-1)}\left(x^{(N)}_3\right) & \text{for } k = N \end{cases} \tag{2.14a}$$

$$\Delta^{(k)j}_3 = \begin{cases} f_3\left([\mathbf{u}]^{(1)}, [T]^{(1)}, [m]^{(1)}\right)\pi^{(1)j}\left(x^{(2)}_3\right) - \sigma_{33}\left(x^{(1)}_3\right)\pi^{(1)j}\left(x^{(1)}_3\right) & \text{for } k = 1 \\ \quad f_3\left([\mathbf{u}]^{(k+1)}, [T]^{(k+1)}, [m]^{(k+1)}\right)\pi^{(k)j}\left(x^{(k+1)}_3\right) - f_3\left([\mathbf{u}]^{(k)}, [T]^{(k)}, [m]^{(k)}\right)\pi^{(k)j}\left(x^{(k+1)}_3\right) & \\ \quad \text{for } k = 2, \ldots, N-1 & \\ \sigma_{33}\left(x^{(N+1)}_3\right)\pi^{(N+1)j}\left(x^{(N+1)}_3\right) - f_3\left([\mathbf{u}]^{(N-1)}, [T]^{(N-1)}, [m]^{(N-1)}\right)\pi^{j(N-1)}\left(x^{(N)}_3\right) & \text{for } k = N \end{cases} \tag{2.14b}$$

and where $\sigma_{iz}|_{z_1}$ and $\sigma_{iz}|_{z_{N+1}}$ are the applied stresses at the top and bottom surfaces of the plate. The resulting local equations of motion can incorporate any general ICMs in a unified fashion without the need for reformulation.

The above mechanical development is a generalization of previous work (Williams, 1999). In particular, the use of the more generalized displacement expansion in Eq. (2.1) has resulted in more general forms for

the equations of motion and the associated boundary conditions. This generalization is reflected as differences in the governing equations compared to those present in previous work. Furthermore, the presence of TDMC results in more complicated functional dependencies in the governing equations. These functional dependencies have been explicitly emphasized in the interfacial jump terms, Eqs. (2.12) and (2.14).

2.4. Energy equations

This portion of the formulation makes use of the pointwise energy equation (Weitsman, 1987)

$$\rho c_v \dot{T} + q_{i,i} - \frac{\partial \sigma_{ij}}{\partial T} T(\dot{\epsilon}_{ij} - \dot{\epsilon}^I_{ij}) - \underline{T\left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right)\dot{m} + \left(\tilde{\mu}_{,i} + T\tilde{s}_{,i} + T_{,i}\tilde{s}\right) f_i} - \eta \sigma_{ij} \dot{\epsilon}^I_{ij} = 0 \tag{2.15}$$

where the strains are given by Eq. (2.5), ρ is the material density, c_v is the specific heat at constant volume, q_i is the heat flux components, $\dot{\epsilon}^I_{ij}$ is the inelastic strain rate, $\tilde{s}$ is the entropy density of the solute, $\tilde{\mu}$ is the internal energy of the solute, f_i is the solute flux components, and η is the partitioning factor for the inelastic work conversion to heat. The underlined terms in Eq. (2.15) represent the effects of solute concentration on the energy. This form for the energy equation is derived using a Helmholtz function which is consistent with displacement and temperature based formulations. The above equation directly introduces coupling effects between the deformations, the temperature, and the solute concentration.

The form of the energy equation without diffusional effects is given by ignoring the underlined terms (Allen, 1991). The corresponding linearly coupled variational principle of the energy equation (obtained by replacing T in the third term with T_R where T_R is some constant reference temperature) is given by (Gilat, 1996). Different variational principles for the energy equation are given by Ben-Amoz (1965), Askar Altay and Cengiz Dokmeci (1996), Li (1992), and Herrman (1963).

Since there do not seem to be any variational principles for the energy equation that incorporate solute effects the method of moments has been used to derive the following system of governing equations for the energy effects

$$\begin{aligned}
&\overline{Q}^a_{\beta,\beta} + \overline{H}^a - \overline{F}^a + \overline{C}^{ja}\dot{\Theta}^j + \sum_{k=1}^{N} \widehat{C}^{(k)ba}\dot{\theta}^{(k)b} - \bar{A}^a + \bar{A}^{aI} - \dot{\overline{W}}^a \\
&\quad \underline{- \overline{B}^{jca}\Theta^j\dot{S}^c - \sum_{k=1}^{N} \breve{B}^{(k)bca}\theta^{(k)b}\dot{S}^c - \sum_{k=1}^{N} \widehat{B}^{(k)jda}\Theta^j\dot{s}^{(k)d} - \sum_{k=1}^{N} \widetilde{B}^{(k)bda}\theta^{(k)b}\dot{s}^{(k)d}} \\
&\quad \underline{+ \overline{P}^a + (\overline{V}^{ja} + \overline{X}^{ja})\Theta^j + \sum_{k=1}^{N} (\widetilde{V}^{(k)ba} + \widetilde{X}^{(k)ba})\theta^{(k)b} + \overline{Y}^{ja}_{\alpha}\Theta^j_{,\alpha} + \sum_{k=1}^{N} \widetilde{Y}^{(k)ba}_{\alpha}\theta^{(k)b}_{,\alpha}} \\
&\quad = 0
\end{aligned} \tag{2.16a}$$

$$\begin{aligned}
&\widehat{Q}^{(k)b}_{\beta,\beta} + \widehat{H}^{(k)b} - \widehat{F}^{(k)b} + \widehat{C}^{(k)ab}\dot{\Theta}^a + \widetilde{C}^{(k)jb}\dot{\theta}^{(k)j} - \widetilde{A}^{(k)b} + \widetilde{A}^{(k)bI} - \dot{\widetilde{W}}^{(k)b} \\
&\quad \underline{- \bar{b}^{(k)acb}\Theta^a\dot{S}^c - \breve{b}^{(k)jcb}\theta^{(k)j}\dot{S}^c - \hat{b}^{(k)adb}\Theta^a\dot{s}^{(k)d} - \tilde{b}^{(k)jdh}\theta^{(k)j}\dot{s}^{(k)d}} \\
&\quad \underline{+ \widetilde{P}^{(k)b} + (\widetilde{V}^{(k)ab} + \breve{X}^{(k)ab})\Theta^a + (\widehat{V}^{(k)jb} + \widehat{X}^{(k)jb})\theta^{(k)j} + \widetilde{Y}^{(k)ba}_{\alpha}\Theta^a_{,\alpha} + \widehat{Y}^{(k)jb}\theta^{(k)j}_{,\alpha}} \\
&\quad = 0
\end{aligned} \tag{2.16b}$$

where the ranges for the superscripts are implied. The definitions used in Eq. (2.16) are given in Appendix A. Again, the underlined terms represent diffusional effects. Upon elimination of the diffusional effects, Eq. (2.16) can be seen to be represent variationally consistent governing energy equations. Eq. (2.16a) is

associated with the global temperature effects, Θ^a, and these equations can be considered to represent "smeared" energy equations for the plate. Eq. (2.16b) are related to the local temperature effects, $\theta^{(k)b}$, and, thus, correspond to the local energy equations. While each of these different types of equations has a particular length scale interpretation it must be noted that these equations exhibit coupling over all of the length scale effects. This is discussed more explicitly in the next section.

The essential thermal boundary conditions are given by the specification of

$$\Theta^a, \quad \theta^{(k)b} \qquad \text{on } \partial\Omega_3 \tag{2.17}$$

The natural thermal boundary conditions on the vertical surfaces of the plate are given by

$$\overline{Q}^r_\alpha = \overline{Q}^{*r}_\alpha, \quad \widehat{Q}^{(k)s}_\alpha = \widehat{Q}^{*(k)s}_\alpha \quad \partial\Omega_4 \tag{2.18}$$

where $\partial\Omega = \partial\Omega_3 + \partial\Omega_4$ and $\overline{Q}^{*r}_\alpha$ and $\widehat{Q}^{*(k)s}_\alpha$ are known flux quantities. It is noted that the thermal boundary conditions, Eqs. (2.17) and (2.18), are consistent with those that would be derived from a variational analysis upon the elimination of the diffusional effects.

The local energy equations, Eq. (2.16b), are coupled through the interfacial constraints on the thermal effects. The temperature jumps due to interfacial discontinuities are given by Eq. (2.10c). The corresponding thermal flux conditions at the interface are

$$q_3^{(k+1)}\Big|_{x_3^{(k+1)}} = q_3^{(k)}\Big|_{x_3^{(k+1)}} = q_3^{(k)I} \tag{2.19}$$

where $q_3^{(k)I}$ is the thermal flux at the kth interface.

A thermal ICM relates the field jumps to the thermal flux across an interface (Boley and Wiener, 1985). The general functional form for such a relation is given by

$$q_3 = e([\mathbf{u}], [T], [m]) \tag{2.20}$$

In general, the thermal ICMs are nonlinear and coupled relations. The response curve of the above relation exhibits similar characteristics as observed in the mechanical ICM, Fig. 2.

The existence of the thermal ICM can be used to simplify the local energy equations. Utilizing Eqs. (2.10c), (2.19) and (2.20) the thermal flux across an interface, $q_3^{(k)I}$, can be expressed directly in terms of the field jumps across an interface. Substituting this result into Eq. (2.16b) gives

$$\begin{aligned}
&\widehat{Q}^{(k)b}_{\beta,\beta} + \delta^{(k)b} - \widehat{F}^{(k)b} + \widehat{C}^{(k)ab}\dot{\Theta}^a + \widetilde{C}^{(k)jb}\dot{\theta}^{(k)j} - \widetilde{A}^{(k)b} + \widetilde{A}^{(k)aI} - \dot{\bar{W}}^{(k)b} - \bar{b}^{(k)acb}\Theta^a\dot{S}^c - \breve{b}^{(k)jcb}\theta^{(k)j}\dot{S}^c \\
&\quad - \hat{b}^{(k)adb}\Theta^a\dot{s}^{(k)d} - \tilde{b}^{(k)jdh}\theta^{(k)j}\dot{s}^{(k)d} + \widetilde{P}^{(k)b} + (\tilde{V}^{(k)ab} + \breve{X}^{(k)ab})\Theta^a + (\widehat{V}^{(k)jb} + \widehat{X}^{(k)jb})\theta^{(k)j} \\
&\quad + \tilde{Y}^{(k)ba}_\alpha\Theta^a_{,\alpha} + \widehat{Y}^{(k)jb}\theta^{(k)j}_{,\alpha} = 0
\end{aligned} \tag{2.21}$$

where

$$\delta^{(k)j} = \begin{cases}
e\left([\mathbf{u}]^{(1)}, [T]^{(1)}, [m]^{(1)}\right)g^{(1)j}\left(x_3^{(2)}\right) - q_3\left(x_3^{(1)}\right)g^{(1)j}\left(x_3^{(1)}\right) & \text{for } k = 1 \\
e\left([\mathbf{u}]^{(k+1)}, [T]^{(k+1)}, [m]^{(k+1)}\right)g^{(k)j}\left(x_3^{(k+1)}\right) - e\left([\mathbf{u}]^{(k)}, [T]^{(k)}, [m]^{(k)}\right)g^{(k)j}\left(x_3^{(k+1)}\right) & \\
\quad \text{for } k = 2, \ldots, N-1 & \\
q_3\left(x_3^{(N+1)}\right)g^{(N+1)j}\left(x_3^{(N+1)}\right) - e\left([\mathbf{u}]^{(N-1)}, [T]^{(N-1)}, [m]^{(N-1)}\right)g^{(N-1)j}\left(x_3^{(N)}\right) & \text{for } k = N
\end{cases} \tag{2.22}$$

where $q_3(x_3^{(1)})$ and $q_3\left(x_3^{(N+1)}\right)$ are the applied thermal fluxes at the top and bottom surfaces of the plate. As was the case for the mechanical ICMs effects, the above form of the governing local energy equations can incorporate any thermal ICM in a unified fashion without any reformulation.

The energy equation portion of the development represents a substantial generalization of existing plate theories that consider the effects of coupled thermal and mechanical fields. This generalization has introduced new types of coupling effects between the fields. These new coupling effects are discussed in more detail below (Section 3).

2.5. Solute conservation equations

The pointwise equation for the conservation of solute (solute diffusion equation) is given by (Weitsman, 1987; Sih et al., 1986)

$$\dot{m} = -f_{i,i} \tag{2.23}$$

While the above equation does not directly exhibit coupling between the deformation, temperature, and solute concentration fields such coupling is implied. The explicit type of coupling between the different fields is introduced by the constitutive relations chosen to related the different fields. This will be discussed in more detail at the end of this section.

Since no variational principle for this equation exists the method of moments is used to obtain the governing equations for the plate theory. Multiplying Eq. (2.23) by L^c and integrating through the thickness of the plate gives

$$\overline{\lambda}^j c\dot{S}^j + \sum_{k=1}^{N} \tilde{\lambda}^{(k)dc}\dot{s}^{(k)d} = -\overline{f}^c_{\alpha,\alpha} - \overline{\omega}^c + \overline{\xi}^c \tag{2.24}$$

where the definitions for the resultants are given in Appendix A. This equation represents the global equations governing the solute diffusion through the plate.

The local plate diffusion equations are obtained by multiplying Eq. (2.23) by $l^{(k)d}$ and integrating through the thickness. After manipulation the following equations are obtained

$$\tilde{\lambda}^d c\dot{S}^d + \sum_{k=1}^{N} \hat{\lambda}^{(k)jd}\dot{s}^{(k)j} = \bar{f}^{(k)d}_{\alpha,\alpha} - \hat{\omega}^{(k)d} + \overline{\xi}^{(k)d} \tag{2.25}$$

Eq. (2.25) is related to the $s^{(k)d}$ terms and can be considered local diffusional equations for the layers in a laminated plate.

The essential thermal boundary conditions for the diffusion component of the theory are given by the specification of

$$S^c, \quad s^{(k)d} \qquad \text{on } \partial\Omega_5 \tag{2.26}$$

The natural diffusional boundary conditions on the vertical surfaces of the plate are given by

$$\overline{f}^c_\alpha = \overline{f}^{*c}_\alpha, \quad \hat{f}^{(k)d}_\alpha = \hat{f}^{*(k)d}_\alpha \quad \partial\Omega_6 \tag{2.27}$$

where $\partial\Omega = \partial\Omega_5 + \partial\Omega_6$ and $\overline{f}^{*c}_\alpha$ and $\hat{f}^{*(k)d}_\alpha$ are known mass flux quantities.

Similar to the previous results, the effects of delaminations on the diffusional response of the plate are incorporated into the theory in a unified fashion through the use of ICMs. The diffusional ICM is functionally given by

$$f_3 = h([\mathbf{u}], [T], [m]) \tag{2.28}$$

The diffusional ICM is in, general, coupled to the mechanical, thermal, and solute effects.

The local diffusional plate equations are simplified using Eq. (2.28) in the same fashion as for the local equations of motion and the local energy equations. The final result is given by

$$\tilde{\lambda}^{d}\, c\dot{S}^{d} + \sum_{k=1}^{N} \hat{\lambda}^{(k)jd}\, \dot{s}^{(k)j} = -\bar{f}^{(k)d}_{\alpha,\alpha} - d^{(k)d} + \overline{\xi}^{(k)d} \tag{2.29}$$

where

$$d^{(k)j} = \begin{cases} h\left([\mathbf{u}]^{(1)}, [T]^{(1)}, [m]^{(1)}\right) l^{(1)j}\left(x_3^{(2)}\right) - f_3\left(x_3^{(1)}\right) l^{(1)j}\left(x_3^{(1)}\right) & \text{for } k = 1 \\ h\left([\mathbf{u}]^{(k+1)}, [T]^{(k+1)}, [m]^{(k+1)}\right) l^{(k)j}\left(x_3^{(k+1)}\right) - h\left([\mathbf{u}]^{(k)}, [T]^{(k)}, [m]^{(k)}\right) l^{(k)j}\left(x_3^{(k+1)}\right) & \\ \quad \text{for } k = 2, \ldots, N-1 & \\ f_3\left(x_3^{(N+1)}\right) l^{(N+1)j}\left(x_3^{(N+1)}\right) - h\left([\mathbf{u}]^{(N-1)}, [T]^{(N-1)}, [m]^{(N-1)}\right) l^{(N-1)j}\left(x_3^{(N)}\right) & \text{for } k = N \end{cases} \tag{2.30}$$

where $f_3\left(x_3^{(1)}\right)$ and $f_3\left(x_3^{(N+1)}\right)$ are the applied mass fluxes at the top and bottom surfaces of the plate. Eqs. (2.25) and (2.30) are completely general unified results that can be used with any form of diffusional ICM without the need for reformulation.

The author is unaware of any previous plate theories that couple solute diffusional effects with the kinematic and thermal behavior in a plate.

2.6. Formulation summary

The fundamental unknowns of the theory are the global displacement terms U^r_α and U^p_3, the local displacement terms $\mu^{(k)s}_\alpha$ and $\mu^{(k)q}_3$, the global temperature terms Θ^a, the local temperature terms $\theta^{(k)b}$, the global solute effects S^c, and the local solute terms $s^{(k)d}$.

The governing system of equations for the kinematic, thermal, and solute unknowns in the theory are the global equations of motion, Eqs. (2.7a) and (2.7b), the local equations of motion, Eq. (2.13), the global and local energy equations, Eqs. (2.16a) and (2.21), and the global and local solute diffusion equations, Eqs. (2.24) and (2.29). The mechanical boundary conditions are given by Eqs. (2.8) and (2.9). The thermal boundary conditions are given by Eqs. (2.17) and (2.18). The corresponding solute diffusion boundary conditions are given by Eqs. (2.26) and (2.27). The interfacial constraints are incorporated in a unified fashion through Eq. (2.14) (for mechanical effects), Eq. (2.22) (for thermal effects), and Eq. (2.30) (for diffusional effects).

The above formulation is independent of any particular set of constitutive relations for the field variables. In general the constitutive relations will exhibit dependencies on all three fundamental fields of the theory. For example, the mechanical behavior of a material might be assumed to be governed by a Hookean type of constitutive model based on superposition of strain effects

$$\sigma_{ij} = C_{ijkl}\left(\epsilon_{kl} - \alpha_{kl}\Delta T - \beta_{kl}\Delta m - \epsilon^I_{kl}\right) \tag{2.31a}$$

where $\alpha_{kl}\Delta T$ and $\beta_{kl}\Delta m$ represent thermal and solution induced strain effects and ϵ^I_{kl} represents the inelastic strain effects due to history-dependent deformations at the material level. Further discussions of mechanical constitutive modeling concepts are given by Malvern (1969), Lemaitre and Chaboche (1990), Christensen (1982), and Khan and Huang (1995) among others. Examples of potential constitutive relations for the thermal and mass fluxes are given by an anisotropic generalization of equations proposed by Weitsman (1987) and Sih et al. (1986)

$$q_i = -K_{ij}^{(1)} T_{,j} - K_{ij}^{(2)} m_{,j} - K_{ij}^{(3)} \epsilon_{kk,j}$$
$$f_i = -F_{ij}^{(1)} T_{,j} - F_{ij}^{(2)} m_{,j} - F_{ij}^{(3)} \epsilon_{kk,j} \tag{2.32}$$

where $K_{ij}^{(n)}$ and $F_{ij}^{(n)}$ are material parameters.

It is noted that the current theory could be extended to the analysis of sublaminates where the "global" component of the field variables are associated with the sublaminate behavior and the local components of the displacement field continue to represent the layer behavior. In this situation, the displacement, temperature, and solute jumps are given by

$$\begin{aligned}
[u_\alpha]^{(\mathrm{SLI})} &= \left(U_\alpha^r P^r(x_3^{(\mathrm{SLI}+1)}) + \mu_\alpha^s p^s(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI}+1)} - \left(U_\alpha^r P^r(x_3^{(\mathrm{SLI}+1)}) + \mu_\alpha^s p^s(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI})} \\
[u_3]^{(\mathrm{SLI})} &= \left(U_3^p \Pi^p(x_3^{(\mathrm{SLI}+1)}) + \mu_3^q \pi^q(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI}+1)} - \left(U_3^p \Pi^p(x_3^{(\mathrm{SLI})}) + \mu_3^q \pi^q(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI})} \\
[T]^{(\mathrm{SLI})} &= \left(\Theta^a G^a(x_3^{(\mathrm{SLI}+1)}) + \theta^b g^b(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI}+1)} - \left(\Theta^a G^a(x_3^{(\mathrm{SLI}+1)}) + \theta^b g^b(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI})} \\
[m]^{(\mathrm{SLI})} &= \left(S^c L^c(x_3^{(\mathrm{SLI}+1)}) + s^d l^d(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI}+1)} - \left(S^c L^c(x_3^{(\mathrm{SLI}+1)}) + s^d l^d(x_3^{(\mathrm{SLI}+1)})\right)^{(\mathrm{SLI})}
\end{aligned} \tag{2.33}$$

where $x_3^{(\mathrm{SLI})}$ are the positions of the sublaminate interfaces. The traction/thermal/mass flux continuity conditions and the ICMs are also applied at these interfaces. Otherwise the formulation remains the same.

3. Theoretical implications

A number of theoretical implications to the currently proposed formulation are discussed in this section.

The first issue to be considered in this section is the existence of coupling between the different length scale effects in the displacement, temperature, and solute fields. The strain–displacement relations, Eq. (2.4), have two different types of coupling effects between the global and local components of the displacement expansions. The global and local components of the expansions appear in a linear fashion in all of the strain components. The inplane strain components introduce products of the global and local displacement terms due to the geometric nonlinearity assumptions. These products occur in the form of mixed products (global and local effects) and nonlinear (just global or just local effects) of both the global and local effects.

As discussed previously the equations of motion, Eqs. (2.7a), (2.7b) and (2.13), can be associated with global and local effects, respectively. However, these equations exhibit a number of strong coupling phenomena between the different length scales in the different fields. Explicitly, linear coupling occurs due to the presence of time derivatives of both the global and local displacement effects in these equations. The coupling phenomena present in the strain–displacement relations are directly introduced into the definitions of the resultants, Eq. (2.7), and, hence, into the equations of motion through the constitutive relations for the layers. These coupling effects take the form of linear, quadratic, and cubic products of the global and local displacement effects. The presence of thermal and solute terms in the constitutive relations introduces several types of coupling. Consideration shows that the different length scale temperature and solute effects appear linearly in the resultants $\overline{N}_{\alpha\beta}^j$, $\overline{R}_{\alpha 3}^j$, $\breve{N}_{3\alpha}^j$, $\breve{R}_{33}^j$, $\widehat{N}_{\alpha\beta}^{(k)j}$, $\widehat{R}_{\alpha 3}^{(k)j}$, $\breve{N}_{3\alpha}^{(k)j}$, and $\breve{R}_{33}^{(k)j}$ (although the weighting of these terms is due to mixed products of the global/local displacement expansion functions and the global/local temperature/solute expansion functions). Nonlinearly coupled terms between the different temperature and solute effects and the displacement effects are also introduced into the equations of motion by the constitutive relations. These effects occur in the form of higher order products of the functions P^r, Π^p, G^a, L^c, $p^{(k)s}$, $\pi^{(k)q}$, $g^{(k)b}$, and $l^{(k)d}$ in the previously mentioned resultants. Further nonlinear coupling effects are introduced by the terms $\breve{M}_{\alpha\beta}^{pj} U_{3,\beta}^p$, $M_{\alpha\beta}^{(k)qj} \mu_{3,\beta}^{(k)q}$, $M_{\alpha\beta}^{(k)jp} U_{3,\beta}^p$, and $\breve{M}_{\alpha\beta}^{(k)jq} \mu_{3,\beta}^{(k)q}$ in the form of

products of the U_i^j, $\mu_i^{(k)j}$, Θ^a, $\theta^{(k)b}$, S^c, and $s^{(k)d}$ as well as higher order products of the expansion functions P^r, Π^p, G^a, L^c, $p^{(k)s}$, $\pi^{(k)q}$, $g^{(k)b}$, and $l^{(k)d}$. Finally, the interfacial terms in the equations of motion; $\Delta_i^{(k)}$, are, in general, coupled functions of the kinematic, thermal, and solute fields in the plate.

The same types of coupling effects between the different global and local fields exist in the governing equations for energy, Eqs. (2.16a) and (2.21), and solute, Eqs. (2.24) and (2.29), as well as in the boundary conditions.

Obviously the above discussion emphasizes the fact that the existence of two characteristic length scales impacts all aspects of the formulation. These coupling effects appear to be unique to the current formulation. The coupling phenomena are important since they can be used to enhance the computational efficiency of the theory while maintaining good accuracy. In particular, Williams (1998, 2000) has shown that accurate predictions for the response of cross ply plates subjected to cylindrical bending can be obtained using a first order (linear in x_3) expansion for the global displacements and third-order expansions for the local expansions. More importantly, these results indicated that the proposed multilength scale theory can be more computationally efficient than a similar fully discrete layer analysis. Increased accuracy for the predictions of the fields within a plate can be obtained by increasing the orders of either the global or local fields (Williams, 1998, 2000). This, in turn, implies that the local history-dependent behavior can (potentially) be modeled with any desired accuracy. The expansions used for the local and global terms in the different fields need not be of the same order or type. The number of potential combinations for these expansions is infinite.

If the theory is linearized (i.e. geometrically linear strains are used) many of the above coupling effects remain.

Associated with the ability to tailor the expansions for the different fields is the ability to transition smoothly through different length scales without any modification of the theory. This fact implies that by appropriate specialization of the displacement, temperature, and solute expansions all currently available displacement (linear or von Karman strain) based, plate theories can be obtained as a subset of the current theoretical framework. This is discussed in the next section.

The next issue that has several important implications is the use of interfacial constitutive models to model the influence of delaminations on the plate response. The theoretical framework has been developed using general functional forms for the ICMs. Thus, any particular ICM can be utilized in the theory without the need for any reformulation. ICMs allow the effects of delaminations to be incorporated into the theory in an internally consistent or unified fashion in terms of the fundamental unknowns of the theory; namely, the local displacement, temperature, and solute effects. No additional unknowns are introduced into the theory by the presence of delaminations and, thus, the computation efficiency is maintained. The use of ICMs does not require any a priori assumptions about the extent or location of cracking/delamination. Therefore, the theory can be used to analyze the initiation, growth, and subsequent interaction of delaminations in plates under in service conditions. Furthermore, ICMs naturally incorporate no interpenetration constraints by requiring that the displacements jumps under compression are zero. In rapid loading situations the ICMs can be implemented in an explicit fashion eliminating the need for iterative solution techniques (as opposed to virtual crack methodologies where iterative schemes are required). Finally, in situations where thin interlaminar regions are expected to significantly influence the response of a laminate ICMs can be used to model the response of these layers without introducing additional layers into the computational effort.

The theory allows for buckling by accounting for geometric nonlinearities. Geometric nonlinearity is introduced through the use of von Karman type assumptions. In general, for composite materials, it is felt that this assumption is sufficiently accurate.

The above capabilities result in a comprehensive, unified theory that can accurately and efficiently analyze the mutual influence of thermo-mechanical-diffusional coupling, delamination, global and local/sub-

laminate buckling, and localized history-dependent mechanisms such as viscoplasticity and damage on the response of laminated or homogeneous plates.

4. Specializations to existing theories

The proposed theory represents a unified framework for the development of different types of plate theories. To illustrate this point this section shows how existing displacement based, variationally consistent plate theories and their thermo-mechanical counterparts (when available) can be obtained via appropriate specializations of the framework.

4.1. Specializations to existing mechanical theories

In the following discussion in this section thermal and diffusional coupling effects are ignored. Thus, the energy equations, Eqs. (2.16a) and (2.21), and the solute conservation equations, Eqs. (2.24) and (2.29), are satisfied identically by assuming $\Theta^a = G^a = \theta^{(k)b} = g^{(k)b} = 0$ and $S^c = L^c = s^{(k)d} = l^{(k)d} = 0$.

Frequently, plate theories assume that there is no though-the-thickness variation in the transverse displacement, i.e. transverse inextensibility. This specialization is obtained from the current framework by using $\Pi^0 = 1$ and $U_3^j = \Pi^j = 0$ for $j > 0$ and $\mu_3^{(k)q} = 0$ for all q.

The simplest types of plate theories are those based on the concept of an equivalent or "smeared" plate. In these theories a laminated plate is replaced by an equivalent plate with smeared properties, i.e. no distinct between lamina is considered. Particular examples of these are considered next.

The simplest "smeared" plate theory is the so-called classical laminated plate theory (CLPT). Reddy (1997) presents the variationally based development of this theory. Whitney (1987) develops the governing equations using the method of moments. The deformation field for this theory is obtained from the current analysis by using the transverse inextensibility restrictions (above) in conjunction with the following restrictions for the inplane displacements; $P^0 = 1$, $P^1 = x_3$, $P^r = 0$ for $r > 1$, $U_\alpha^1 = \partial_\alpha u_3 = -\partial_\alpha U_3^0$, and $\mu_\alpha^{(k)s} = p^{(k)s} = 0$ for all s. In order to obtain the correct forms of the governing equations it must be recognized that the linear terms in the inplane displacement expansions do not represent independent variables. The necessary manipulations of Eqs. (2.7a) and (2.7b) to obtain the CLPT governing equations are given in Appendix B. A wide variety of results generated with this theory (and the correlations with exact solutions where applicable) are given by Reddy (1997) and Whitney (1987).

The simplest "smeared" transverse shear deformation theory is first order shear deformation theory (FSDT). The variationally derived linear and von Karman based strain measure formulations of this theory are presented by Reddy (1997). The corresponding vectorial or method of moments developments are presented by Whitney (1987). This theory can be obtained from the current framework by using the above restrictions for transverse inextensibility and the following restrictions on the inplane displacements u_α; $\mu_\alpha^{(k)s} = p^{(k)s} = 0$ for all s and $P^0 = 1$ and $P^2 = x_3$. The appropriate forms of the governing equations given in the above references directly correspond to the equations provided by the current theory. Extensive results obtained from this theory for static bending, buckling, and vibration and the correlation with exact solutions are given by Reddy (1997).

The next family of "smeared" plate theories are the third order theories. This type of theory represents the lowest order of the so-called higher order shear deformation theories (HSDT). One of the first proposed theories of this type was developed by Lo et al. (1977a,b). The specializations required to obtain the theory of Lo et al. are $P^0 = 1$, $P^1 = x_3$, $P^2 = x_3^2$, $P^3 = x_3^3$ and $P^r = 0$ for $r > 3$, $p^{(k)s} = 0$ for all s, $\Pi^0 = 1$, $\Pi^1 = x_3$, $\Pi^2 = x_3^2$, and $\Pi^p = 0$ for $p > 2$, and $\pi^{(k)q} = 0$ for all q. The governing equations obtained in the theory of Lo

et al. directly correspond to the equations provided by current theory. Results comparing the predictive capabilities of this third order theory with exact results are given by Lo et al.

Reddy (1989, 1990) shows how all theories up to and including third order, displacement based, theories can be obtained from a single third order theory through appropriate change of variables. Geometric nonlinearity is introduced through the use of von Karman type strain measures. The appropriate specializations of the current theory are given by (using Reddy's notation); $P^0 = 1$, $P^1 = x_3$, $U^1_\alpha = U^0_{3,\alpha}$, $P^2 = x_3 - \frac{4}{3h^2}x_3^3$, $U^2_\alpha = \beta\hat{\phi}_\alpha$, $P^3 = \frac{1}{2}x_3^2 - \lambda\psi_{3,\alpha}$, $P^4 = \frac{4}{3h^2}x_3^3$, $U^4_\alpha = -\beta w_{s,\alpha}$, $P^5 = \frac{1}{3}x_3^3$, $U^5_\alpha = -\gamma\theta_{3,\alpha}$, $\Pi^0 = 1$, $U^0_3 = \beta w_s$, $\Pi^1 = 1$, $U^1_3 = \alpha w_b$, $\Pi^2 = x_3$, $U^2_3 = \lambda\psi_3$, $\Pi^2 =$, $U^3_3 = \lambda\theta_3$, $\mu^{(k)s}_\alpha = p^{(k)s} = 0$, and $\mu^{(k)q}_3 = \pi^{(k)q} = 0$. The constants α, β, λ, and γ are tracers used to reduce the general third order formulation to specific theories. The necessary manipulations required to obtain Reddy's equations (1990) are given in Appendix C.

A more advanced type of theory than the "smeared" plate theories are the "zig-zag" theories. These theories satisfy traction and displacement continuity conditions in an a priori fashion. A partial review and discussion of some of the limitations of this type of theory is given by Averill and Yip (1996). A nonlinear (in the von Karman sense) version of this type of theory is given by Di Sciuva (1986). To obtain Di Sciuva's theory from the current formulation the specializations given by constant through-the-thickness transverse displacement are employed. The following specializations are also used $P^0 = 1$, $P^1 = x_3 + \sum_{k=1}^{N-1} a^{(k)}_\alpha (x_3 - x^{(k)}_3) H(x_3 - x^{(k)}_3)$ where $H(x_3 - x^{(k)}_3)$ is the Heaviside function, $a^{(k)}_\alpha$ (Eq. (6) of Di Sciuva (1986)) are constants related to the relative properties of the lamina, and $\mu^{(k)s}_\alpha = p^{(k)s} = 0$ for the inplane displacements are used. The governing equations given by Di Sciuva are obtained directly from Eqs. (2.7a) and (2.7b). Other versions of "zig-zag" theories can be obtained in a similar manner.

Soldatos (1995) has used the method of moments/vectorial formulation to develop a general, variationally consistent, "smeared" plate theory. The theory is based on the use of linear strains. He has shown how simplification of this theory can be used to obtain existing variationally consistent "smeared" plate theories. To obtain Soldatos' variationally consistent formulation from the current theory the following specializations are used; $p^{(k)s} = \mu^{(k)s}_\alpha = \pi^{(k)q} = \mu^{(k)q}_3 = 0$, $P^1 = x_3$, $U^1_\alpha = -U^0_{3,\alpha}$, and the other global expansion functions correspond to the functions ϕ_{2i} and ϕ_{2i+1} discussed by Soldatos (1995).

Discrete layer theories (Reddy, 1987; Williams and Addessio, 1997) represent one of the most accurate types of laminated plate theories. These theories directly model the response of each lamina in a plate and, subsequently, couple them through interfacial constraints. Reddy (1997) summarizes existing discrete layer theories, discusses how his discrete layer theory can be specialized to obtain existing linear theories, and provides results obtained from his formulation. Reddy's discrete layer displacement based theory can be obtained from the current formulation by using $P^r = 0$ and $\Pi^p = 0$ and using any desired form for the $p^{(k)s}$ and $\pi^{(k)q}$ subject to the requirement that $p^{(k)s}(x_3 = 0) = \pi^{(k)q}(x_3 = 0) = 0$. The identification of the resulting equivalence of the governing equations given by Reddy (1987) and Eqs. (2.6a) and (2.6b) is readily apparent.

While the development thrust of the theories of Soldatos (1995) and Reddy (1987) were concerned primarily with "smeared" or discrete layer theories, respectively, consideration of the theories shows that they are, in fact, equivalent "single length scale" theories. In particular, the displacement fields in both theories can be associated with the behavior of either a layer within a laminate or with the entire laminate (but not both simultaneously). The resulting analysis is dependent on the chosen length scale. For example, by identifying Soldatos' functions ϕ as associated with a lamina/layer length scale and introducing appropriate interfacial constraints then the theory proposed by Reddy (1987) can be obtained. Soldatos (1993) discusses in more detail the modifications of his theory necessary to obtain the Lagrange function based, layerwise variant of Reddy's theory. Conversely, identifying Reddy's layerwise functions ϕ and ψ as associated with the entire laminate thickness and eliminating the interfacial constraints then Soldatos' formulation (1995) can be obtained. These theories cannot consider multilength scale effects. It is worth noting that the re-identification of length scales necessary to convert the theories of Reddy and Soldatos to either discrete or

"smeared" analyses are not necessary within the current theoretical framework since the current theory considers the pertinent length scales in a unified fashion.

The final type of mechanical theory considered in this section are the so-called "variable kinematic (VK) models" (Reddy, 1997). A summary of the literature for these types model is presented by Reddy (1997). Reddy's VK model can be obtained from the current theoretical framework by ignoring coupled terms in the governing equations given above (i.e. by assuming that the global governing equations are only functions of the terms in the global portion of the displacement expansion and the local governing equations are only functions of the local displacement expansion terms). The resulting equations are coupled (as discussed by Reddy) by eliminating the displacements at some number (given by the order of the global theory used) of arbitrary nodes for the local theory.

The above discussion has not exhaustively addressed the necessary specializations required to obtain every currently available, mechanical plate theory. However, the outlined specializations have, in most cases, considered the most general plate theories of different classes currently available. The relationships between these different general theories and simpler plate theories are outlined in the cited references. In the cases where specific theories where considered the relation to other theories in the associated class of theories is clear and, thus, the necessary specializations required to obtain these theories are simple variations of the outlined specializations. In light of the above comments it can be seen that any currently available displacement based, linear or von Karman strain based, variationally consistent, mechanical plate theory can be obtained as a specialization of the current framework.

4.2. Specializations to existing thermo-mechanical theories

In the following discussion it is assumed that diffusional effects are null. This specialization is obtained by using $S^c = L^c = s^{(k)d} = l^{(k)d} = 0$.

A review of work carried out that has considered the influence of specified temperature fields on the flexure, buckling, and vibration response of both laminated and homogeneous plates is given by Tauchert (1991). The results of the referenced works can be obtained by using the same functional form for the global temperature field terms (Θ^a and G^a) as given in a particular work, requiring that the local temperature effects be zero ($\theta^{(k)b} = g^{(k)b} = 0$), and ignoring the energy portion of the current formulation (no evolution of the temperature field). It is noted that the governing equations of these types of studies directly correspond to the above mechanical theories where temperature is considered within the context of the constitutive relations and superposition of strain effects is used (c.f. Jonnalagadda et al., 1993). Thus, the relations between these types of theories and the current framework is evident in light of the relations given in Section 4.1.

A recent work by Praveen and Reddy (1998) on the response of functionally graded plates is an example of a one-way coupled analysis. In this work the coupling effects between the mechanical and thermal responses in the energy equation are ignored. Within the current theory this amounts to using

$$\overline{Q}^a_{\beta,\beta} + \overline{H}^a = 0 \tag{4.1}$$

as the energy equation. This equation is solved for the temperature distribution within the plate. The temperature field (which does not evolve) is subsequently used in the analysis of the deformation of the plate. The mechanical aspect of work is based on Reddy's third order mechanical theory (see above).

A fully coupled thermo-mechanical analysis has been developed by Gilat and Aboudi (1996) for cylindrical bending of laminated plates. The lamina material behavior was modeled using Hookean constitutive relations and superposition of strain effects. The mechanical component of the work is based on the von Karman version of Reddy's third order theory (1984, 1990) for zero transverse normal stress and accounts for the presence of initial geometric imperfections. The specializations of the displacement field, Eq. (2.1),

required to obtain this theory are given by $P^0 = 1$, $P^1 = x_3 - \beta k_1 x_3^3$, $P^2 = \beta k_1 x_3^3$, $U_\alpha^2 = -U_3^0$, $P^j = 0$ for $j \geqslant 3$, $\Pi^0 = 1$, $U_3^0 = U_3^0 + U_{30}$ $\Pi^j = 0$ for $j \geqslant 1$, $p^{(k)s} = \mu_\alpha^{(k)s} = 0$, and $\pi^{(k)q} = \mu_3^{(k)q} = 0$ where U_{30} is the initial (known) geometric imperfection in the plate. The zero transverse normal stress assumption used in the development of the equilibrium equations of the theory requires that $\breve{R}_{33}^j = 0$ be used directly in Eq. (2.7b) and that plane stress reduced stiffnesses be used in the evaluation of the constitutive relations. The cylindrical bending assumptions require that differentiation with respect to x_2 be zero. Subject to these restrictions the discussion concerning the development of a general third order plate theory is valid. The temperature field used by Gilat and Aboudi is initially assumed to have the form

$$\Delta T(x_\alpha, x_3, t) = \Theta^0 + \Theta^1 x_3 + \Theta^2 x_3^2 + \Theta^3 x_3^3 \tag{4.2}$$

Satisfying either heat flux or the temperature boundary conditions on the top and bottom surfaces in an a priori fashion leads to a temperature field of the form

$$\Delta T(x_\alpha, x_3, t) = \Theta^0\left(1 + \tau_2 x_3^2\right) + \Theta^1\left(x_3 + \tau_4 x_4^3\right) + \tau_1 x_3^2 + \tau_3 x_4^4 \tag{4.3}$$

where τ_i are known functions of the boundary conditions. The pertinent forms of the energy equations obtained by Gilat and Aboudi where derived using the method of moments. The assumptions of zero transverse normal strain and stress in conjunction with a Hookean constitutive relation can be used to obtain

$$\dot{\epsilon}_{33}^I = \left(\frac{C_{\alpha\beta 33}}{C_{3333}}\right)\left(\dot{\epsilon}_{\alpha\beta} - \alpha_{\alpha\beta}\dot{T} - \dot{\epsilon}_{\alpha\beta}^I\right) - \alpha_{33}\dot{T} \tag{4.4}$$

Using these results the relationship

$$C_{ijkl}\alpha_{kl}T\left(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^I\right) = T\delta_{\alpha\beta}\left(\dot{\epsilon}_{\alpha\beta} - \dot{\epsilon}_{\alpha\beta}^I\right) + \delta_{33}^2 T\dot{T} \tag{4.5}$$

is obtained. The δ_{ij} are relations between the material properties and the coefficient of thermal expansion obtained from the constitutive relation for the normal transverse stress. Substituting Eq. (4.5) into the definitions of $\overline{a}^a$ and $\overline{a}^{aI}$ shows that these resultant terms are functions of the U_i^j and $\dot{T}$ (i.e. the functional relation is obtained $\overline{a}^a - \overline{a}^{aI} = b_1^a(U_\alpha^j) + b_2^a\dot{T}_0 + b_3^a\dot{T}_1$). Using the following identifications between the terms in the energy equations defined by Gilat and Aboudi and the current framework

$$\begin{aligned}
G11 &= \overline{C}^{00} + b_2^0 \\
G12 = G21 &= \overline{C}^{10} + b_3^0 = \overline{C}^{01} + b_2^1 \\
G22 &= \overline{C}^{11} + b_3^1 \\
S_1 &= \overline{Q}_{1,1}^0 + \overline{h}^0 - \overline{f}^0 + b_1^0 - \dot{\overline{W}}^0 \\
S_2 &= \overline{Q}_{1,1}^1 + \overline{h}^1 - \overline{f}^1 + b_1^1 - \dot{\overline{W}}^1
\end{aligned} \tag{4.6}$$

the equivalence between the energy equations becomes apparent.

A fully coupled thermoelastic discrete layer analysis has been developed by Baczynski (1991) for the perfectly bonded plates. The analysis is derived using a theory based on internal constraints which is consistent with a variational analysis. The variation through the thickness of the fields in the individual layers is assumed to be given by first order Lagrange polynomials. Thus, the mechanical portion of the theory is equiv-

alent to Reddy's discrete layer theory (1987) and the previous discussion showing the relations between Reddy's theory and the current framework apply to Baczynski's theory. The energy portion of Baczynski's theory is based on the energy equation given in terms of internal energy rather than a Helmholtz form of the energy equation. However, using the relations between internal energy and the Helmholtz free energy ($u = \psi - Ts$ where u is the internal energy, ψ is the Helmholtz free energy, T is the temperature, and s is the entropy) and the functional relations $\psi = \phi(\epsilon_{ij}, T)$ and $s = s\ (\epsilon_{ij}, T)$ relating the Helmholtz free energy and the entropy to the strain and temperature fields Baczynski's energy equations for the plate can be directly reduced to Eq. (2.21). See Allen (1991) for a concise discussion of the procedure. The following identifications between Baczynski's resultants and the current theory's resultants can subsequently be made

$$\begin{aligned} \underline{\underline{Q}}_{(a)} &= \overline{\underline{\underline{Q}}}^{a} \\ q_{(a)} &= \overline{F}^{a} \\ \delta^{l}_{(a)} &= \overline{H}^{a} \\ \underline{\underline{T}}_{(a)(b)} \cdot \nabla\dot{\psi} - \dot{\epsilon}_{(a)} &\equiv \overline{C}^{ja}\dot{\Theta}^{j} + \sum_{k=1}^{N} \widehat{C}^{(k)ba}\dot{\theta}^{(k)b} - \bar{A}^{a} \end{aligned} \tag{4.7}$$

5. Summary and conclusion

A new type of plate theory based on a generalized formulation utilizing a distinctive multilength scale analysis that incorporates the effects of thermo-mechanical-diffusional coupling, delamination, and geometric nonlinearity has been presented. The ability to consider this range of effects simultaneously seems to be unique to the present theory. The inimitable multilength scale aspects of the theory are obtained through the use of generalized two length scale expansions for the displacement, temperature, and solute fields. The functional forms and orders of the expansions at both length scales are arbitrary subject to the constraint that the terms in the different length scale expansions be independent functions. The theory incorporates delamination and/or nonlinear elastic or inelastic interfacial behavior for the mechanical, thermal, and concentration effects in a unified fashion through the use of generalized interfacial constitutive (cohesive) relations. The use of ICMs allows the theory to analyze the influence of delamination initiation, growth, and interaction in plates without introducing any a priori assumptions about the location or extent of the delaminations. The resulting unified theoretical framework represents a new type/class of plate theory that can be used to consider the general thermo-diffusionally-mechanically coupled response of laminated (or homogeneous) plates in the presence of delaminations, buckling, and/or nonlinear material behavior.

It has been shown that appropriate specialization of the proposed framework can be used to obtain currently available variationally derived, displacement-based plate theories capable of carrying out both mechanical and thermo-mechanical analysis of laminated plates.

While the present paper has focused on the theoretical formulation of a unified framework for the development of plate theories capable of carrying out coupled field analyses, it is useful to give some (speculative) consideration to the potential for numerical implementation of the full theory. As the applications of laminated structures become more demanding it can be expected that coupling different aspects of the mechanical, thermal, and diffusional response characteristics of plates will become necessary. Given the complicated physics of such problems any tools capable of carrying out such analyses can be expected to be equally complex. The numerical implementation of such theories can consequentially be expected to require significant effort. The proposed theory, which seems to represent one of the few tools available for such analyses, certainly would require this level of effort for implementation. From a computational

efficiency perspective, as mentioned in Section 3, previous work (Williams, 1999, 2000) on the mechanical behavior of plates indicates that the use of the multiscale analysis approach can enhance computational efficiency as compared to single scale approaches by requiring fewer unknowns to achieve a given level of solution accuracy. Current (unpublished) work on the mechanical behavior of laminated plates is indicating that the computational savings (as compared to traditional finite element approaches) obtained from the proposed approach for the behavior of plates subjected to static loadings continues to hold for dynamic loading situations. At this point it can be hoped that similar results can be obtained for the coupled field analysis represented by the current theory. Future work will consider the numerical implementation of the proposed theory and the resulting computational requirements.

Appendix A. Resultant definitions

The following definitions have been employed in Eq. (2.7).

$$\overline{I}^{mn} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \rho P^m P^n \,\mathrm{d}x_3, \quad \widehat{I}^{(k)mn} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho p^{(k)m} P^n \,\mathrm{d}\bar{x}_3, \quad \breve{I}^{mn} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \rho \Pi^m \Pi^n \,\mathrm{d}x_3$$

$$\breve{I}^{(k)mn} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho \pi^{(k)m} \Pi^n \,\mathrm{d}\bar{x}_3, \quad \widetilde{I}^{(k)mn} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho p^{(k)m} p^{(k)n} \,\mathrm{d}\bar{x}_3, \quad I^{*(k)mn} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho \pi^{(k)m} \pi^n \,\mathrm{d}\bar{x}_3$$

$$\overline{N}^m_{\alpha\beta} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{\alpha\beta} P^m \,\mathrm{d}x_3, \quad \breve{N}^m_{\alpha\beta} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{\alpha\beta} \Pi^m \,\mathrm{d}x_3$$

$$\widehat{N}^{(k)m}_{\alpha\beta} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{\alpha\beta} p^{(k)m} \,\mathrm{d}\bar{x}_3, \quad \breve{N}^{(k)m}_{\alpha\beta} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{\alpha\beta} \pi^{(k)m} \,\mathrm{d}\bar{x}_3$$

$$\overline{R}^m_{\alpha 3} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{\alpha 3} \partial_3 P^m \,\mathrm{d}x_3, \quad \breve{R}^m_{33} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{33} \partial_3 \Pi^m \,\mathrm{d}x_3$$

$$\widehat{R}^{(k)m}_{\alpha 3} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{\alpha 3} \partial_3 p^{(k)m} \,\mathrm{d}\bar{x}_3, \quad \breve{R}^{(k)m}_{33} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{33} \partial_3 \pi^{(k)m} \,\mathrm{d}\bar{x}_3$$

$$\breve{M}^{mn}_{\alpha\beta} = t \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{\alpha\beta} \Pi^m \Pi^n \,\mathrm{d}x_3, \quad M^{(k)mn}_{\alpha\beta} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{\alpha\beta} \pi^{(k)m} \Pi^n \,\mathrm{d}\bar{x}_3, \quad \breve{M}^{(k)mn}_{\alpha\beta} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{\alpha\beta} \pi^{(k)m} \pi^{(k)n} \,\mathrm{d}\bar{x}_3$$

$$\overline{F}^m_{\alpha} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} f_\alpha P^m \,\mathrm{d}x_3, \quad \breve{F}^m_3 = \int_{x_3^{(1)}}^{x_3^{(N+1)}} f_3 \Pi^m \,\mathrm{d}x_3$$

$$\widehat{F}^{(k)m}_{\alpha} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} f_\alpha p^{(k)m} \,\mathrm{d}x_3, \quad \breve{F}^{(k)m}_3 = \int_{x_3^{(k)}}^{x_3^{(k+1)}} f_3 \pi^{(k)m} \,\mathrm{d}x_3$$

$$\overline{\tau}_\alpha^m = t_\alpha P^m\big|_{x_3^{(N+1)}} + t_\alpha P^m\big|_{x_3^{(1)}} = \sigma_{\alpha 3} P^m\big|_{x_3^{(1)}}^{x_3^{(N+1)}}$$

$$\breve{\tau}_3^m = t_3 \Pi^m\big|_{x_3^{(N+1)}} + t_3 \Pi^m\big|_{x_3^{(1)}} = \sigma_{33} \Pi^m\big|_{x_3^{(1)}}^{x_3^{(N+1)}}$$

$$\hat{\tau}_\alpha^{(k)m} = t_\alpha p^{(k)m}\big|_{x_3^{(k+1)}} + t_\alpha p^{(k)m}\big|_{x_3^{(k)}} = \sigma_{\alpha 3} p^{(k)m}\big|_{x_3^{(k)}}^{x_3^{(k+1)}}$$

$$\check{\tau}_3^{(k)m} = t_3 \pi^{(k)m}\big|_{x_3^{(k+1)}} + t_3 \pi^{(k)m}\big|_{x_3^{(k)}} = \sigma_{33} \pi^{(k)m}\big|_{x_3^{(k)}}^{x_3^{(k+1)}}$$

where the appropriate ranges for the superscripts m and n are implied. It is emphasized that the terms incorporating the $p^{(k)s}$ and $\pi^{(k)q}$ exist only within the kth layer.

The following definitions have been used in Eq. (2.9)

$$\overline{T}_\alpha^j = \int_{x_3^{(1)}}^{x_3^{(N+1)}} t_i P^j \, \mathrm{d}x_3, \quad \breve{T}_3^j = \int_{x_3^{(1)}}^{x_3^{(N+1)}} t_i \Pi^j \, \mathrm{d}x_3$$

$$\hat{T}_\alpha^{(k)j} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} t_i p^{(k)j} \, \mathrm{d}\bar{x}_3, \quad \hat{T}_3^{(k)j} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} t_i \pi^{(k)j} \, \mathrm{d}\bar{x}_3$$

where again the appropriate ranges for the superscripts are implied.

The following definitions have been employed in Eq. (2.16)

$$\overline{Q}_\alpha^a = \int_{x_3^{(1)}}^{x_3^{(N+1)}} q_\alpha G^a \, \mathrm{d}x_3, \quad \hat{Q}_\alpha^{(k)b} = \int_{z_k}^{x_3^{(k+1)}} q_\alpha g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\overline{H}^a = q_3 G^a\big|_{x_3^{(1)}}^{x_3^{(N+1)}}, \quad \hat{H}^{(k)b} = q_3 g^{(k)b}\big|_{x_3^{(k)}}^{x_3^{(k+1)}}$$

$$\overline{F}^a = \int_{x_3^{(1)}}^{x_3^{(N+1)}} q_3 \partial_3 G^a \, \mathrm{d}x_3, \quad \hat{F}^{(k)b} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} q_3 \partial_3 g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\overline{C}^{ja} = \int_{x_3^{(1)}}^{x_3(N+1)} \rho c_v G^j G^a \, \mathrm{d}x_3, \quad \hat{C}^{(k)ba} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho c_v g^{(k)j} G^a \, \mathrm{d}\bar{x}_3, \quad \widetilde{C}^{(k)ja} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \rho c_v g^{(k)j} g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\bar{A}^a = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \frac{\partial \sigma_{ij}}{\partial T} T \dot{\epsilon}_{ij} G^a \, \mathrm{d}x_3, \quad \widetilde{A}^{(k)b} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \frac{\partial \sigma_{ij}}{\partial T} T \dot{\epsilon}_{ij} g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\bar{A}^{aI} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \frac{\partial \sigma_{ij}}{\partial T} T \dot{\epsilon}_{ij}^I G^a \, \mathrm{d}x_3, \quad \widetilde{A}^{(k)bI} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \frac{\partial \sigma_{ij}}{\partial T} T \dot{\epsilon}_{ij}^I g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\dot{\overline{W}}^a = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \sigma_{ij} \dot{\epsilon}_{ij}^I G^a \, \mathrm{d}x_3, \quad \dot{\widetilde{W}}^{(k)b} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \sigma_{ij} \dot{\epsilon}_{ij}^I g^{(k)b} \, \mathrm{d}\bar{x}_3$$

$$\overline{B}^{jca} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) G^j L^c G^a \,\mathrm{d}x_3, \quad \breve{B}^{(k)bca} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) g^{(k)b} L^c G^a \,\mathrm{d}\bar{x}_3$$

$$\widehat{B}^{(k)jda} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) G^j l^{(k)d} G^a \,\mathrm{d}\bar{x}_3, \quad \tilde{B}^{(k)bda} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) g^{(k)b} l^{(k)d} G^a \,\mathrm{d}\bar{x}_3$$

$$\overline{P}^a = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \tilde{\mu}_{,i} f_i G^a \,\mathrm{d}x_3$$

$$\overline{V}^{ja} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \tilde{s}_{,i} f_i G^j G^a \,\mathrm{d}x_3, \quad \widetilde{V}^{(k)ba} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s}_{,i} f_i g^{(k)b} G^a \,\mathrm{d}\bar{x}_3$$

$$\overline{Y}^{ja} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \tilde{s} f_\alpha G^j G^a \,\mathrm{d}x_3, \quad \widetilde{Y}^{(k)ba} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s} f_\alpha g^{(k)b} G^a \,\mathrm{d}\bar{x}_3$$

$$\overline{X}^{ja} = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \tilde{s} f_3 \partial_3 G^j G^a \,\mathrm{d}x_3, \quad \widetilde{X}^{(k)ba} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s} f_3 \partial_3 g^{(k)b} G^a \,\mathrm{d}\bar{x}_3$$

$$\overline{b}^{(k)acb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) G^a L^c g^{(k)b} \,\mathrm{d}\bar{x}_3, \quad \breve{b}^{(k)jcb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) g^{(k)j} L^c g^{(k)b} \,\mathrm{d}\bar{x}_3$$

$$\hat{b}^{(k)adb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) G^a l^{(k)d} g^{(k)b} \,\mathrm{d}\bar{x}_3, \quad \tilde{b}^{(k)jdb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \left(\tilde{s} + \frac{\partial \tilde{\mu}}{\partial T}\right) g^{(k)j} l^{(k)d} g^{(k)b} \,\mathrm{d}\bar{x}_3$$

$$\widetilde{P}^{(k)b} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{\mu}_{,i} f_i g^{(k)b} \,\mathrm{d}\bar{x}_3$$

$$\widehat{V}^{(k)jb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s}_{,i} f_i g^{(k)j} g^{(k)b} \,\mathrm{d}\bar{x}_3$$

$$\widehat{Y}_\alpha^{(k)jb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s} f_\alpha g^{(k)j} g^{(k)b} \,\mathrm{d}\bar{x}_3$$

$$\breve{X}^{(k)ab} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s} f_3 \partial_3 G^a g^{(k)b} \,\mathrm{d}\bar{x}_3, \quad \widehat{X}^{(k)jb} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \tilde{s} f_3 \partial_3 g^{(k)j} g^{(k)b} \,\mathrm{d}\bar{x}_3$$

The following definitions have been employed in Eqs. (2.24) and (2.25)

$$\overline{\lambda}^j c = \int_{x_3^{(1)}}^{x_3^{(N+1)}} L^j L^c \,\mathrm{d}x_3, \quad \tilde{\lambda}^{(k)dc} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} l^{(k)d} L^c \,\mathrm{d}\bar{x}_3, \quad \hat{\lambda}^{(k)jd} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} l^{(k)j} l^{(k)d} \,\mathrm{d}\bar{x}_3$$

$$\bar{f}^c_\alpha = \int_{x_3^{(1)}}^{x_3^{(N+1)}} f_\alpha L^c \,\mathrm{d}x_3, \quad \hat{f}^{(k)d}_\alpha = \int_{x_3^{(k)}}^{x_3^{(k+1)}} f_\alpha l^{(k)d} \,\mathrm{d}\bar{x}_3$$

$$\overline{\omega}^c = f_3 L^c\big|_{x_3^{(1)}}^{x_3^{(N+1)}}, \quad \hat{\omega}^{(k)d} = f_3 l^{(k)d}\big|_{x_3^{(k)}}^{x_3^{(k+1)}}$$

$$\overline{\xi}^c = \int_{x_3^{(1)}}^{x_3^{(N+1)}} \partial_3 L^c \,\mathrm{d}x_3, \quad \hat{\xi}^{(k)d} = \int_{x_3^{(k)}}^{x_3^{(k+1)}} \partial_3 l^{(k)d} \,\mathrm{d}\bar{x}_3$$

Appendix B. Classical laminated plate theory

This appendix outlines the necessary manipulations of Eqs. (2.7a) and (2.7b) required to obtain the correct form of the governing equations for the CLPT. Body forces are ignored for this analysis. Direct application of Eqs. (2.7a) and (2.7b) gives

$$\begin{aligned} \overline{I}^{rj}\ddot{U}^r_\alpha &= \overline{N}^j_{\alpha\beta,\beta} - \overline{R}^j_{\alpha 3} + \overline{\tau}^j_\alpha \\ \breve{I}^{00}\ddot{U}^0_3 &= \breve{N}^0_{3\beta,\beta} - \breve{R}^0_{33} + \partial_\alpha\left(\breve{M}^{00}_{\alpha\beta}U^0_{3,\beta}\right) + \breve{\tau}^0_3 \end{aligned} \tag{B.1}$$

The correct forms of the inplane governing equations for CLPT are obtained by realizing that $\partial_3 P^0 = 0$. This result gives $\overline{R}^j_{\alpha z} = 0$ and Eq. (B.1a) directly reduces to the following form

$$\overline{I}^{rj}\ddot{U}^r_\alpha = \overline{N}^j_{\alpha\beta,\beta} + \overline{\tau}^j_\alpha \tag{B.2}$$

The correct bending equation can be obtained as follows. Substituting the relations $U^1_\alpha = -U^0_{3,\alpha}$, $\overline{R}^1_{\alpha 3} = \overline{N}^0_{\alpha 3}$, and $\breve{M}^{10}_{\alpha\beta} = \breve{M}^{01}_{\alpha\beta} = \overline{N}^1_{\alpha\beta}$ into Eq. (B.1a) for $j = 1$ and differentiating with respect to x_α gives

$$\breve{N}^0_{3\beta,\beta} = -\overline{I}^{01}\ddot{U}^0_{\alpha,\alpha} + \overline{I}^{11}\ddot{U}^0_{3,\alpha\alpha} + \breve{M}^{01}_{\alpha\beta,\alpha\beta} + \overline{\tau}^1_{\alpha,\alpha} \tag{B.3}$$

Substituting these relations into Eq. (B.1b) and using the assumption of plane stress gives the final form of the governing bending equation for the theory.

$$\breve{I}^{00}\ddot{U}^0_3 + \overline{I}^{01}\ddot{U}_{\alpha,\alpha} - \overline{I}^{11}\ddot{U}^0_{3,\alpha\alpha} = \breve{M}^{01}_{\alpha\beta,\alpha\beta} + \partial_\alpha\left(\breve{N}^0_{3\beta}U^0_{3,\beta}\right) + \breve{\tau}^0_3 \tag{B.4}$$

where the relations $\breve{M}^{00}_{\alpha\beta} = \breve{N}^0_{3\beta}$ has been used.

Appendix C. Generalized third order laminated plate theory

The displacement field proposed by Reddy (1990) for the development of the generalized third order laminated plate theory is given by

$$\begin{aligned} u_\alpha &= U^0_\alpha - x_3\beta w_{\mathrm{b},\alpha} + \beta\left(x_3 - \frac{4}{3h^2}x_3^3\right)\hat{\phi}_\alpha - \frac{1}{2}x_3^2\lambda\psi_{3,\alpha} - \frac{4}{3h^2}x_3^3\beta w_{\mathrm{s},\alpha} - \frac{1}{3}x_3^3\gamma\theta_{3,\alpha} \\ u_3 &= \beta w_{\mathrm{s}} + \alpha w_{\mathrm{b}} + x_3\lambda\psi_3 + x_3^2\gamma\theta_3 \end{aligned} \tag{C.1}$$

Reddy defines the resultants used in his equilibrium equations in the following fashion

$$\left(N_{ij}, M_{ij}, P_{ij}, S_{ij}\right) = \int_{-h^2}^{h^2} \left(1, x_3, x_3^2, x_3^3\right)\sigma_{ij}\,\mathrm{d}x_3 \tag{C.2}$$

where contracted notation has been employed.

Using the previously given inplane displacement expansion functions in the current framework gives the following equilibrium equations

$$\begin{aligned}
&\overline{N}^0_{\alpha\beta,\beta} - \overline{R}^0_{\alpha 3} = 0\\
&\overline{N}^1_{\alpha\beta,\beta} - \overline{R}^1_{\alpha 3} = 0\\
&\overline{N}^2_{\alpha\beta,\beta} - \overline{R}^2_{\alpha 3} = 0\\
&\overline{N}^3_{\alpha\beta,\beta} - \overline{R}^3_{\alpha 3} = 0\\
&\overline{N}^4_{\alpha\beta,\beta} - \overline{R}^4_{\alpha 3} = 0\\
&\overline{N}^5_{\alpha\beta,\beta} - \overline{R}^5_{\alpha 3} = 0
\end{aligned} \tag{C.3}$$

The corresponding out of plane equilibrium equations are

$$\begin{aligned}
&\breve{N}^0_{3\beta,\beta} - \breve{R}^0_{33} + \partial_\alpha\left(\overline{N}^0_{\alpha\beta}(\beta w_\mathrm{s} + \gamma w_\mathrm{b})\right) + \breve{\tau}^0_3 = 0\\
&\breve{N}^1_{3\beta,\beta} - \breve{R}^1_{33} + \partial_\alpha\left(\overline{N}^0_{\alpha\beta}(\beta w_\mathrm{s} + \gamma w_\mathrm{b})\right) + \breve{\tau}^1_3 = 0\\
&\breve{N}^2_{3\beta,\beta} - \breve{R}^2_{33} = 0\\
&\breve{N}^2_{3\beta,\beta} - \breve{R}^2_{33} = 0
\end{aligned} \tag{C.4}$$

where the identity $\breve{M}^{00}_{\alpha\beta} = \overline{N}_{\alpha\beta}$ has been used. Note that the restrictions imposed by Reddy requiring that the only nonlinearities retained are related to w_b and w_s, that there are no shear effects on the top and bottom surfaces of the plate, and that the normal stress effects on the top and bottom surfaces of the plate are related only to w_b and w_s have been directly employed in the above equations.

Consideration of Eq. (C.3a) shows that these equations are equivalent to Reddy's Eq. (18a-1) and (18a-2) since $\overline{R}^0_{\alpha 3} = 0$ and the $\overline{N}^0_{\alpha\beta}$ in the current theory are equivalent to the $N_{\alpha\beta}$ in Reddy's theory.

Similarly consideration of Eq. (C.3c) shows that these equations are Eqs. (18a-5) and (18a-6) since the following identities hold

$$\begin{aligned}
\overline{N}^2_{\alpha\beta} &= \beta\left(M_{\alpha\beta} - \frac{4}{3h^2}S_{\alpha\beta}\right)\\
\overline{R}^2_{\alpha 3} &= \beta\left(N_{\alpha 3} - \frac{4}{h^2}P_{\alpha 3}\right)
\end{aligned} \tag{C.5}$$

The equivalence between the out-of-plane equilibrium equations in the current theory and Reddy's theory can be demonstrated as follows. Eqs. (C.3b) are related to $w_{\mathrm{b},\alpha}$ while Eq. (C.4b) are related to w_b. To obtain consistency Eq. (C.3b) is differentiated with respect to x_α and the result is subtracted from Eq. (C.4b) to give

$$\overline{N}^1_{\alpha\beta,\alpha\beta} + \breve{\tau}^0_3 + \partial_\alpha\left(\overline{N}^0_{\alpha\beta}(\beta w_\mathrm{s} + \gamma w_\mathrm{b})\right) = 0$$

This process is equivalent to that used to develop CLPT equations. The following identities

$$\overline{N}^{1}_{\alpha\beta} = \alpha M_{\alpha\beta}$$
$$\breve{\tau}^{0}_{3} = \alpha q$$

show the required equivalence of the equilibrium equations for w_b between the two theories.

Differentiating Eq. (C.3e) with respect to x_α and subtracting the result from Eq. (C.4b) gives

$$\overline{R}^{4}_{\alpha 3,\alpha} - \overline{N}^{4}_{\alpha\beta,\alpha\beta} + \breve{N}^{1}_{\alpha 3,\alpha} + \breve{\tau}^{1}_{3} + \partial_\alpha\left(\overline{N}^{0}_{\alpha\beta}(\beta w_s + \gamma w_b)\right) = 0$$

Identifying the following relations

$$\overline{N}^{4}_{\alpha\beta} = \beta\frac{4}{3h^2}S_{\alpha\beta}$$
$$\overline{R}^{4}_{\alpha 3} = \beta\frac{4}{h^2}P_{\alpha\beta}$$
$$\breve{N}^{1}_{\alpha 3} = \beta N_{\alpha 3}$$

shows the required equivalence of the equilibrium equations for w_s.

Differentiating Eq. (C.3d) with respect to x_α and subtracting the result from Eq. (C.4c) gives

$$\overline{N}^{3}_{\alpha\beta,\alpha\beta} - \breve{R}^{2}_{33} = 0$$

The relations

$$\overline{N}_{\alpha\beta} = 2\lambda P_{\alpha\beta}$$
$$\breve{R}^{2}_{33} = N_{33}$$

in conjunction with the above equilibrium equation demonstrate the equivalence of the equilibrium equations for ψ_3 between the two theories.

Carrying out similar processes using Eqs. (C.3f) and (C.4d) also indicates that Reddy's equilibrium equations for θ_3 can be obtained from the current theory.

The above manipulations indicate that Reddy's generalized, third order theory can be obtained consistently from the current theoretical framework.

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